An Achievability Bound for Type-Based Unsourced Multiple Access

Deekshith Pathayappilly Krishnan*, Kaan Okumus*, Khac-Hoang Ngo†, Giuseppe Durisi*
*Department of Electrical Engineering, Chalmers University of Technology, 41296 Gothenburg, Sweden
Email: {deepat, okumus, durisi}@chalmers.se
†Department of Electrical Engineering (ISY), Linköping University, 58183 Linköping, Sweden
Email: khac-hoang.ngo@liu.se

***Abstract*—We derive an achievability bound to quantify the performance of a type-based unsourced multiple access system—an information-theoretic model for grant-free multiple access with correlated messages. The bound extends available achievability results for the per-user error probability in the unsourced multiple access framework, where, different from our setup, message collisions are treated as errors. Specifically, we provide an upper bound on the total variation distance between the type (i.e., the empirical probability mass function) of the transmitted messages and its estimate over a Gaussian multiple access channel. Through numerical simulations, we illustrate that our bound can be used to determine the message type that is less efficient to transmit, because more difficult to detect. We finally show that a practical scheme for type estimation, based on coded compressed sensing with approximate message passing, operates approximately 3 dB away from the bound, for the parameters considered in the paper.**

I. Introduction

Next generation wireless communication systems should support accurate decision-making from distributed network data [1]. Typically, decision-making is performed at a central node, to which data-collecting entities (e.g., Internet of Things (IoT) sensors) transmit their information. The central node is often interested in a function of the received data. For instance, applications like over-the-air aggregation in wireless federated learning [2], point cloud transmission [3], and majority-vote computation [4] involve estimating the type, i.e., the empirical probability mass function (PMF) of the received data. In this paper, we establish a performance bound for communication systems employing type-based decision-making. Specifically, we generalize the information-theoretic analysis of the unsourced multiple-access (UMA) model presented in [5] to the case of type-based unsourced multiple-access (TUMA).

The UMA framework is relevant for massive IoT connectivity. In UMA, all transmitters share a common codebook as well as the communication medium (e.g., an additive white Gaussian noise channel), and the receiver produces a list of the transmitted codewords. In [5], an achievability bound on the per-user error probability is obtained under the assumption that the active transmitters select their message uniformly at random. In the analysis, the event that two or more users select the same codeword (message collision) is modeled as an error. Indeed, the probability that this occurs is typically negligible for the parameters considered in [5] (hundreds of active users and 2^{100} messages). On the contrary, in this paper, we are interested in the case in which multiple users transmit the same message and the receiver is tasked with estimating the set of transmitted messages along with their multiplicities, i.e., the number of users sending each message.

Type-based multiple access was originally introduced in [6] in the context of parameter estimation from distributed data. The TUMA framework has been recently introduced in [7], in the context of multi-target tracking. There, the authors analyze the performance of different compressed-sensing-inspired algorithms for type estimation. However, no performance bound is derived. This work addresses such a gap.

Contributions: We derive a numerically computable upper bound on the total variation distance between the transmitted message type and the estimated type for the TUMA model. Our analysis relies on some of the tools used in [5], such as Chernoff's bound and Gallager's ρ-trick. The novel challenge in the TUMA setting arises from the presence of additional error events, such as the incorrect estimation of message multiplicities, which make the parametrization and partitioning of error events more complex. In contrast, in the UMA case, the error events can be straightforwardly parametrized by the number of misdetected or impostor (i.e., false-positive) messages, enabling a simple partitioning of the error events and a subsequent application of Gallager's ρ-trick. In the TUMA setting, the message multiplicity errors make the application of Gallager's ρ-trick more nuanced.

We use our bound to numerically characterize, for a selected family of TUMA message types, and for a fixed number of active users, the minimum energy per bit (E_{b}/N_0) required to achieve a target estimation error, as a function of a distance metric between the considered type and an UMA uniform type. We measure this distance in terms of total variation. Our analysis reveals that the required E_{b}/N_0 is maximized for intermediate distance values and exceeds the E_{b}/N_0 required in the UMA case. However, as the distance approaches one, the E_{b}/N_0 reduces significantly, and drops below the value required for UMA, due to the reduced number of multiplicities to detect and the higher power allocated per transmitted message. Finally, we present an adaptation to TUMA of the coded compressed sensing with approximate message passing (CCS-AMP) scheme proposed in [8]. The gap between the E_{b}/N_0 value predicted by our bound and the value required by the proposed CCS-AMP scheme is around 3 dB.

Notation: We denote scalar system parameters by uppercase non-italic letters, scalar random variables by uppercase italic letters, deterministic scalars by lowercase italic letters, vectors by uppercase boldface italic letters, and deterministic vectors by lowercase boldface italic letters, e.g., $\mathrm{K_a}$, X, x, $\boldsymbol{X}$, and $\boldsymbol{x}$, respectively. We let I_N be the $\mathrm{N}\times\mathrm{N}$ identity matrix and $\mathbf{0}$ be the all-zero vector. We denote the N-dimensional Euclidean space by $\mathbb{R}^\mathrm{N}$, the set of naturals as $\mathbb{N}$, $\mathbb{N}\cup\{0\}$ as $\mathbb{N}_0$, and its N-fold product as $\mathbb{N}_0^\mathrm{N}$. We set $[m:n]=\{m,\dots,n\}$, $m\leq n$, and m, $n\in\mathbb{N}$; if $m=1$, $[m:n]=[n]$. For $x\in\mathbb{R}$, $\lceil x\rceil$ is its ceiling value. For $\boldsymbol{x}\in\mathbb{R}^\mathrm{N}$ and $\mathcal{S}\subseteq[\mathrm{N}]$, $\boldsymbol{x}_\mathcal{S}$ denotes the restriction of the vector to $\mathcal{S}$; its dimension is $|\mathcal{S}|$. For a vector $\boldsymbol{n}$, $\mathtt{Supp}(\boldsymbol{n})$ is the set of its non-zero entries. We denote the Gaussian distribution with mean $\mathbf{0}$ and covariance matrix A by $\mathcal{N}(\mathbf{0},\mathsf{A})$. We use $\|\cdot\|$ for the ℓ_2-norm and $\|\cdot\|_1$ for the ℓ_1-norm. We use $\sim$ to specify the distribution of a random variable, $\perp\!\!\!\perp$ for independence, and $\stackrel{D}{=}$ for equality in distribution. We denote the indicator function as $\mathbb{1}\{\cdot\}$, and the total variation distance as $\mathbb{TV}(\cdot,\cdot)$. Unless stated otherwise, all logarithms are natural.

II. System Model

We consider a scenario wherein a large number of transmitters (users), of which only $\mathrm{K_a}$ are *active*, communicate with a common receiver, over N uses of a real-valued Gaussian multiple access channel (GMAC). We assume that $\mathrm{K_a}$ is fixed and known to the receiver. The transmitters communicate over N channel uses. Such devices could be, for instance, sensors deployed to track the states of certain targets. The total number of states could potentially be large and each state corresponds to a message chosen by a transmitter. We denote the total number of messages by M. However, we assume that only $\mathrm{M_a}$ out of M messages are active, and the active message set is denoted as $\mathcal{M}_\mathrm{a}$. In a target-tracking scenario, the active messages could correspond to target locations within a small region near the sensors. We assume that each user chooses a message from $\mathcal{M}_\mathrm{a}$. This results in a type over $[\mathrm{M_a}]$, to be defined shortly. We assume $\mathrm{M_a}\ll\mathrm{M}$ and $\mathrm{M_a}\leq\mathrm{K_a}$, e.g., $\mathrm{M_a}=10^1$, $\mathrm{K_a}=10^2$, and $\mathrm{M}=10^4$. Furthermore, we assume that $\mathrm{M_a}$ is known but $\mathcal{M}_a$ is unknown to the receiver.

The channel output is given by

$$\boldsymbol{Y}=\sum_{k=1}^{\mathrm{K_a}}\boldsymbol{x}_k+\boldsymbol{Z} \tag{1}$$

where $\boldsymbol{Z}\sim\mathcal{N}(\mathbf{0},\mathsf{I}_\mathrm{N})$ is the N-dimensional noise vector, independent of the channel input vector $\boldsymbol{x}_k$ of user k, $k\in[\mathrm{K_a}]$, and $\boldsymbol{Y}$ is the channel output. We assume that the channel inputs satisfy a maximum power constraint

$$\|\boldsymbol{x}_k\|^2\leq\mathrm{NP},\ k\in[\mathrm{K_a}]. \tag{2}$$

In our setting, multiple users may transmit the same message to the receiver. For instance, in the multi-target tracking problem, multiple sensors might report the same location of a tracked target. We utilize an $\mathrm{M_a}$-sparse vector in $\mathbb{N}_0^\mathrm{M}$ to describe the list of transmitted/decoded messages and their multiplicities. We refer to this vector as a multiplicity vector. Specifically, let $\boldsymbol{n}=[n_1,\dots,n_\mathrm{M}]^\mathrm{T}$ be the multiplicity vector corresponding to the messages transmitted by the users. If we denote by W_k the message chosen by user k, we have that

$$n_m=\sum_{k=1}^{\mathrm{K_a}}\mathbb{1}\{W_k=m\}. \tag{3}$$

We have $\|\boldsymbol{n}\|_1=\mathrm{K_a}$ and its support

$$\mathtt{Supp}(\boldsymbol{n})=\{m\in[\mathrm{M}]:n_m>0\} \tag{4}$$

has cardinality $\mathrm{M_a}$. For the UMA case [5], $n_m=1$ for all transmitted messages, and $\|\boldsymbol{n}\|_1=|\mathtt{Supp}(\boldsymbol{n})|$.

The decoder observes $\boldsymbol{Y}$ and obtains an estimate $\hat{\boldsymbol{N}}=[\hat{N}_1,\dots,\hat{N}_\mathrm{M}]^\mathrm{T}$ of $\boldsymbol{n}$. We assume that the decoder's estimate is subject to a constraint on the average total variation distance. Specifically, given the types $\boldsymbol{t}=\boldsymbol{n}/\mathrm{K_a}$ and $\hat{\boldsymbol{T}}=\hat{\boldsymbol{N}}/\mathrm{K_a}$, the decoder's estimate must satisfy $\mathbb{E}\left[\mathbb{TV}(\boldsymbol{t},\hat{\boldsymbol{T}})\right]\leq\epsilon$, where the expectation is taken over the distributions of the channel input and the noise. Furthermore, $\epsilon\in(0,1)$ is pre-specified, and

$$\mathbb{TV}(\boldsymbol{t},\hat{\boldsymbol{T}})=\frac{1}{2}\sum_{m=1}^{\mathrm{M}}|t_m-\hat{T}_m|. \tag{5}$$

We shall refer to the setting just introduced as the TUMA setting. Now, we provide a formal definition of a code for this setting. To this end, it is convenient to define the set of multiplicity vectors on $[\mathrm{M}]$ as

$$\mathfrak{P}=\left\{\boldsymbol{n}\in\mathbb{N}_0^\mathrm{M}:|\mathtt{Supp}(\boldsymbol{n})|=\mathrm{M_a},\ \|\boldsymbol{n}\|_1=\mathrm{K_a}\right\}.$$

Definition 1 (*TUMA Code*)**.** *An $(\mathrm{M},\mathrm{N},\mathrm{P},\epsilon)$ TUMA code for a GMAC with $\mathrm{K_a}$ active users and $\mathrm{M_a}$ active messages resulting in the message multiplicity vector $\boldsymbol{n}$, consists of an encoder $\xi:[\mathrm{M}]\mapsto\mathbb{R}^\mathrm{N}$ that produces a channel input satisfying (2) and a decoder $\psi:\mathbb{R}^\mathrm{N}\mapsto\mathfrak{P}$ that produces a multiplicity vector based on the output $\boldsymbol{Y}$ of the GMAC, and satisfies the constraint $\mathbb{E}[\mathbb{TV}(\boldsymbol{n}/\mathrm{K_a},\psi(\boldsymbol{Y})/\mathrm{K_a})]\leq\epsilon$.*

III. A Random-Coding Bound for TUMA

Our main result is a random-coding achievability bound stated in the next theorem.

Theorem 1 (*Random-Coding Bound*)**.** *For a GMAC with $\mathrm{K_a}$ active users and $\mathrm{M_a}$ active messages, and a fixed message multiplicity vector $\boldsymbol{n}$ with $|\mathtt{Supp}(\boldsymbol{n})|=\mathrm{M_a}$ and $\|\boldsymbol{n}\|_1=\mathrm{K_a}$, there exists a $(\mathrm{M},\mathrm{N},\mathrm{P},\epsilon)$ TUMA code for which*

$$\epsilon\leq\sum_{t=1}^{\mathrm{K_a}}\frac{t}{\mathrm{K_a}}\tilde{p}_t+p_0+p_1. \tag{6}$$

Here,

$$p_0=\mathrm{M_a}\,\mathbb{P}\left[\|\boldsymbol{C}_1\|^2>\mathrm{NP}\right] \tag{7}$$

with $\boldsymbol{C}_1\sim\mathcal{N}(\mathbf{0},\mathrm{P}'\mathsf{I}_\mathrm{N})$, and $0<\mathrm{P}'<\mathrm{P}$,

$$p_1=e^{-\frac{\mathrm{N}\delta^2}{8}} \tag{8}$$

with

$$\delta \in \left(0, \frac{1}{1-2\delta^*_{\min}} - 1\right) \tag{9}$$

and

$$\delta^*_{\min} = \frac{1}{2} - \min_{t,\mathcal{S},\ell,i,j,\rho} \lambda^* \tag{10}$$

where λ^* *defined in* (21)*. Finally,*

$$\tilde{p}_t = \sum_{\ell=0}^{\mathrm{M_a}} \sum_{\mathcal{S}\in\mathfrak{S}_\ell} \sum_{i=0}^{\mathrm{M_a}-\ell} \sum_{\mathcal{N}\in\mathfrak{N}_i} \sum_{j=\ell}^{t-\mathrm{M_a}+\ell+i} e^{\rho \mathrm{N}R - \mathrm{N}E_0}. \tag{11}$$

In both (10) *and* (11)*,* $\rho \in [\rho_{\min}, 1]$ *for a pre-fixed* $\rho_{\min} \in (0,1)$*, and* t*,* $\mathcal{S}$*,* ℓ*,* i*,* j *in* (10) *are optimized over the range of values given in* (6) *and* (11)*,*

$$\mathfrak{S}_\ell = \{\mathcal{S} \subseteq [\mathrm{M_a}] : |\mathcal{S}| = \ell, \|\boldsymbol{n}_\mathcal{S}\|_1 \le t\} \tag{12}$$

$$\mathfrak{N}_i = \{\mathcal{N} \subseteq [\mathrm{M_a}] \setminus \mathcal{S} : |\mathcal{N}| = i, \|\boldsymbol{n}_\mathcal{N}\|_1 \ge t + i - \|\boldsymbol{n}_\mathcal{S}\|_1\}. \tag{13}$$

Furthermore,

$$R = \frac{1}{\mathrm{N}} \log \mathrm{M}' \tag{14}$$

$$E_0 = \frac{1}{2}\log(1-2b\rho) + \rho a \tag{15}$$

$$\mathrm{M}' = \frac{\mathrm{M}^\ell}{\ell!}\binom{j-1}{\ell-1}\binom{t-j-1}{\mathrm{M_a}-\ell-i-1}\min\{\mathrm{M}_0^+, \mathrm{M}^+\} \tag{16}$$

$$\mathrm{M}_0^+ = \binom{t - \|\boldsymbol{n}_\mathcal{S}\|_1 + i - 1}{i-1} \tag{17}$$

$$\mathrm{M}^+ = \binom{\|\boldsymbol{n}_\mathcal{S}\|_1 + \|\boldsymbol{n}_\mathcal{N}\|_1 - t - 1}{i-1} \tag{18}$$

$$a = \frac{1}{2}\log(1 + 2\mathrm{P}' c_{\min}\lambda^*) \tag{19}$$

$$b = \lambda^*\left(1 - \frac{1}{1+2c_{\min}\mathrm{P}'\lambda^*}\right) \tag{20}$$

$$\lambda^* = \frac{(c_{\min}\mathrm{P}'-2) + \sqrt{(c_{\min}\mathrm{P}'-2)^2 + 4c_{\min}\mathrm{P}'(1+\rho)}}{4c_{\min}\mathrm{P}'(1+\rho)} \in \left(0, \frac{1}{2}\right) \tag{21}$$

$$c_{\min} = \left\lceil \frac{\|\boldsymbol{n}_\mathcal{S}\|_1^2}{\ell} \right\rceil + \left\lceil \frac{(t - \|\boldsymbol{n}_\mathcal{S}\|_1)^2}{i} \right\rceil + \left\lceil \frac{(t-j)^2}{\mathrm{M_a}-\ell-i} \right\rceil + \left\lceil \frac{j^2}{\ell} \right\rceil. \tag{22}$$

The proof of the theorem has a structure similar to the proof of [5, Theorem 1], in which an UMA scenario is considered. Specifically, it relies on random coding, Chernoff's bound, and the use of Gallager's ρ-trick [9, Eq. (2.28)]. The use of Gallager's ρ-trick is more delicate in the TUMA case. Indeed, in the UMA setting, the error events, i.e., the misdetection of some messages and the inclusion of impostor messages, are such that the number of misdetected messages coincides with the number of impostor messages. This enables a convenient partitioning of the error event space, achieved by parametrizing the error events through the number of misdetected (or impostor) messages. This, in turn, facilitates the application of Gallager's ρ-trick.

In TUMA, however, the space of error events is much larger. Indeed, a TUMA decoder can incorrectly estimate the multiplicities of correctly decoded messages, either inflating or deflating them. As a consequence, not even the sum of multiplicities of the misdetected messages coincides with that of the impostor messages. This necessitates a more nuanced parametrization of the error events, which needs to account for both the number and the position of the misdetected messages, impostor messages, and inflated and deflated detected messages to effectively partition the error event space. Using this novel parametrization, we upper-bound the probability of error events in each partition by analyzing its Chernoff's exponent, conditioned on a high-probability set. This leads to the term p_1 in (8). Furthermore, to reduce complexity, we apply Gallager's ρ-trick only once, unlike in [5], where it is applied twice.

One final challenge is that the chosen performance metric, i.e., the total variation distance, is a nonlinear function of the multiplicity vectors. This nonlinearity complicates the estimation of the total number of error events contributing to the union bound summation in Gallager's ρ-trick. While in the UMA case this number can be estimated using a straightforward counting argument, our estimation involves linearizing the total variation distance and leveraging expressions for integer-valued solutions to partition equations (see Lemma 4).

Now, we present the proof of Theorem 1.

Proof of Theorem 1. We present the proof in multiple steps. First, we state some supporting lemmas. The proof of these lemmas are delegated to the appendices.

In Lemma 1 below, we prove that the ℓ_1 distance between the transmitted multiplicity vector and the decoded multiplicity vector is even and does not exceed $2\mathrm{K_a}$.

Lemma 1. *Fix any two multiplicity vectors* $\boldsymbol{n}$ *and* $\hat{\boldsymbol{n}}$ *such that* $\|\boldsymbol{n}\|_1 = \|\hat{\boldsymbol{n}}\|_1 = \mathrm{K_a}$*,* $\boldsymbol{n} \ne \hat{\boldsymbol{n}}$*,* $\mathtt{Supp}(\boldsymbol{n}) \subset [\mathrm{M}]$*,* $\mathtt{Supp}(\hat{\boldsymbol{n}}) \subset [\mathrm{M}]$*,* $|\mathtt{Supp}(\boldsymbol{n})| = |\mathtt{Supp}(\hat{\boldsymbol{n}})| = \mathrm{M_a}$ *and* $\mathrm{K_a} \ge \mathrm{M_a}$*. Then,* $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$*, for some* $t \in [\mathrm{K_a}]$*.*

Proof. See Appendix A. □

In Lemma 2 below, we solve an optimization problem that will turn out to be useful in analyzing the Chernoff's exponent.

Lemma 2. *Let* $\rho \in (0,1]$*,* $\mathrm{P}' > 0$*, and* $c > 0$*. Define*

$$E_0'(\lambda) = \frac{\rho}{2}\log(1 + 2\mathrm{P}'c\lambda) + \frac{1}{2}\log\left(1 - 2\rho\lambda\left(1 - \frac{1}{1+2\mathrm{P}'c\lambda}\right)\right) \tag{23}$$

where $\lambda > 0$ *satisfies*

$$1 - 2\rho\lambda\left(1 - \frac{1}{1+2\mathrm{P}'c\lambda}\right) > 0. \tag{24}$$

Then E_0' *is maximized at* λ^**, and* $\lambda^* < 1/2$*.*

Proof. See Appendix B. □

In Lemma 3 below, we prove that a certain function is monotonically decreasing. We will use this result to upper-bound an exponential term arising from Chernoff's bound.

Lemma 3. *For* $\mathrm{N} \in \mathbb{Z}_+$ *and* $\delta \in (0,1)$, *let* $\boldsymbol{z} \in \mathbb{R}^{\mathrm{N}}$ *be such that* $\|\boldsymbol{z}\|^2 \leq \mathrm{N}(1+\delta)$. *Furthermore, let* $\rho \in (0,1]$, $\mathrm{P}' > 0$, *and* λ^* *as in* (21). *Define*

$$f(c) = \lambda^* \left(\|\boldsymbol{z}\|^2 - \frac{\|\boldsymbol{z}\|^2}{1+2c\mathrm{P}'\lambda^*} \right) - \frac{\mathrm{N}}{2}\log(1+2c\mathrm{P}'\lambda^*)\,. \tag{25}$$

Then, for $c > 0$, $f(c)$ *is monotonically decreasing in* c.

Proof. See Appendix C. □

Finally, we present a lemma in which we bound the number of multiplicity vectors at a given ℓ_1 distance from the transmitted multiplicity vector. Furthermore, we provide a lower-bound on the ℓ_2 distance between any decoded multiplicity vector and the transmitted multiplicity vector.

Lemma 4. *Fix a multiplicity vector* $\boldsymbol{n}$ *such that* $\|\boldsymbol{n}\|_1 = \mathrm{K_a}$, $|\mathtt{Supp}(\boldsymbol{n})| = \mathrm{M_a}$, $\mathtt{Supp}(\boldsymbol{n}) \subset [\mathrm{M}]$, *and* $\mathrm{K_a} \geq \mathrm{M_a}$. *Fix* $t \in [\mathrm{K_a}]$. *Let* $\hat{\boldsymbol{n}}$ *denote a generic multiplicity vector such that* $\|\hat{\boldsymbol{n}}\|_1 = \mathrm{K_a}$, $|\mathtt{Supp}(\hat{\boldsymbol{n}})| = \mathrm{M_a}$, $\mathtt{Supp}(\hat{\boldsymbol{n}}) \subset [\mathrm{M}]$. *Furthermore, let* $\mathcal{S} = \mathtt{Supp}(\boldsymbol{n}) \cap \mathtt{Supp}(\hat{\boldsymbol{n}})^c$, $|\mathcal{S}| = \ell$, $\|\boldsymbol{n}_{\mathcal{S}}\|_1 \geq t$, $\widehat{\mathcal{S}} = \mathtt{Supp}(\boldsymbol{n})^c \cap \mathtt{Supp}(\hat{\boldsymbol{n}})$, $|\widehat{\mathcal{S}}| = \ell$, $\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{S}}}\|_1 = j$,

$$\mathcal{N} = \{m \in \mathtt{Supp}(\boldsymbol{n}) \cap \mathtt{Supp}(\hat{\boldsymbol{n}}) : n_m \geq \hat{n}_m\} \tag{26}$$

with $|\mathcal{N}| = i$, *and* $\widehat{\mathcal{N}} = \mathtt{Supp}(\boldsymbol{n}) \cap \mathtt{Supp}(\hat{\boldsymbol{n}}) \cap \mathcal{N}^c$. *Then, for fixed* ℓ, $\mathcal{S}$, i, $\mathcal{N}$, *and* j, *the total number of vectors* $\hat{\boldsymbol{n}}$ *satisfying* $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$ *is utmost* M', *where* M' *is defined in* (16). *Furthermore, such an* $\hat{\boldsymbol{n}}$ *satisfies* $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|^2 \geq c_{\min}$, *with* $c_{\min}$ *defined in* (22).

Proof. See Appendix D. □

The proof of the above lemma involves linearizing the equation $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$, and counting the integer valued solutions of the resulting partition equations to obtain M'.

1) Codebook Generation: For $m \in [\mathrm{M}]$ and $\mathrm{P}' < \mathrm{P}$, we generate the codewords $\boldsymbol{C}_{\mathrm{m}} \sim \mathcal{N}(\boldsymbol{0}, \mathrm{P}'\mathsf{I}_{\mathrm{N}})$ independently. To send message W_k, the transmitter chooses codeword $\boldsymbol{C}_{W_k}$ and set $\boldsymbol{X}_k = \mathbb{1}\left\{\|\boldsymbol{C}_{W_k}\|^2 \leq \mathrm{NP}\right\}\boldsymbol{C}_{W_k}$.

2) Decoder: The decoder outputs the multiplicity vector $\hat{\boldsymbol{N}} \in \mathfrak{P}$ minimizing $\|\boldsymbol{Y} - \boldsymbol{C}(\hat{\boldsymbol{N}})\|^2$, where

$$\boldsymbol{C}(\hat{\boldsymbol{N}}) = \sum_{m \in \mathtt{Supp}(\hat{\boldsymbol{N}})} \hat{N}_m \boldsymbol{C}_m. \tag{27}$$

Note that the probability that a tie occurs is zero.

Next, we analyze the average total variation between $\boldsymbol{t} = \boldsymbol{n}/\mathrm{K_a}$ and $\hat{\boldsymbol{T}} = \hat{\boldsymbol{N}}/\mathrm{K_a}$ where the average is with respect to the noise vector $\boldsymbol{Z}$ in (1), and the random codebook.

3) Error Analysis: Similar to [5], we replace the joint probability distribution over which $\mathbb{E}[\mathbb{TV}(\boldsymbol{t}, \hat{\boldsymbol{T}})]$ is computed with the distribution for which $\boldsymbol{X}_k \overset{D}{=} \boldsymbol{C}_{W_k}$. This can be done at the expense of adding an additional total variation term (the p_0 term in (7)). Next, we evaluate $\mathbb{E}[\mathbb{TV}(\boldsymbol{t}, \hat{\boldsymbol{T}})]$ conditioned on the *high probability event* $\mathcal{A} = \left\{\|\boldsymbol{Z}\|^2 \leq \mathrm{N} + \mathrm{N}\delta\right\}$, where δ is chosen as in (9). This yields an additional penalty of p_1 defined in (8), which is obtained by using the concentration bound for the sum of chi-squared random variables given in [10, Eq. (2.19)]. From Lemma 1, we can express $\mathbb{E}\left[\mathbb{TV}(\boldsymbol{t}, \hat{\boldsymbol{T}})\mathbb{1}\{\boldsymbol{Z} \in \mathcal{A}\}\right]$ as

$$\mathbb{E}\left[\mathbb{TV}(\boldsymbol{t}, \hat{\boldsymbol{T}})\mathbb{1}\{\boldsymbol{Z} \in \mathcal{A}\}\right] = \sum_{t=1}^{\mathrm{K_a}} \frac{t}{\mathrm{K_a}} p_t \tag{28}$$

where $p_t = \mathbb{P}[\|\boldsymbol{n} - \hat{\boldsymbol{N}}\|_1 = 2t, \boldsymbol{Z} \in \mathcal{A}]$. Next, we will upper-bound p_t.

Without loss of generality, we assume $\mathcal{M}_{\mathrm{a}} = [\mathrm{M_a}]$. We define the set of all decoded message multiplicity vectors at an ℓ_1 distance of $2t$ from $\boldsymbol{n}$ as

$$\hat{\Pi}_t(\boldsymbol{n}) = \{\hat{\boldsymbol{n}} : \|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t\}\,. \tag{29}$$

Set $\boldsymbol{C}(\boldsymbol{n}) = \sum_{m \in \mathtt{Supp}(\boldsymbol{n})} n_m \mathbf{C}_m$, $c(\hat{\boldsymbol{n}}) = \|\boldsymbol{n} - \hat{\boldsymbol{n}}\|^2$, and

$$\mathcal{E}(\hat{\boldsymbol{n}}) = \left\{\|\boldsymbol{Z} + \sqrt{\mathrm{P}'c(\hat{\boldsymbol{n}})}\boldsymbol{X}'\| < \|\mathbf{Z}\|\right\}. \tag{30}$$

We observe that

$$p_t = \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \{\text{Decoded vector is } \hat{\boldsymbol{n}}\} \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right] \tag{31}$$

$$= \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \{\|\boldsymbol{Y} - \boldsymbol{C}(\hat{\boldsymbol{n}})\| < \|\boldsymbol{Y} - \boldsymbol{C}(\boldsymbol{n})\|\} \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right] \tag{32}$$

$$= \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \{\|\boldsymbol{Z} + \boldsymbol{C}(\boldsymbol{n}) - \boldsymbol{C}(\hat{\boldsymbol{n}})\| < \|\boldsymbol{Z}\|\} \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right] \tag{33}$$

$$= \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \left\{\|\boldsymbol{Z} + \sqrt{\mathrm{P}'\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|^2}\boldsymbol{X}'\| < \|\boldsymbol{Z}\|\right\} \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right] \tag{34}$$

$$= \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \left\{\|\boldsymbol{Z} + \sqrt{\mathrm{P}'c(\hat{\boldsymbol{n}})}\boldsymbol{X}'\| < \|\boldsymbol{Z}\|\right\} \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right] \tag{35}$$

$$= \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}}) \cap \{\boldsymbol{Z} \in \mathcal{A}\}\right]. \tag{36}$$

where (33) follows from the fact that $\boldsymbol{Y} = \boldsymbol{C}(\boldsymbol{n}) + \boldsymbol{Z}$, and (34) follows by noting that $\boldsymbol{C}(\boldsymbol{n}) - \boldsymbol{C}(\hat{\boldsymbol{n}}) \perp\!\!\!\perp \boldsymbol{Z}$,

$$\boldsymbol{C}(\boldsymbol{n}) - \boldsymbol{C}(\hat{\boldsymbol{n}}) \overset{\mathrm{D}}{=} \sqrt{\mathrm{P}'\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|^2}\boldsymbol{X}' \tag{37}$$

where $\boldsymbol{X}' \sim \mathcal{N}(\mathbf{0}, \mathsf{I}_{\mathrm{N}})$, and $\boldsymbol{X}' \perp\!\!\!\perp \boldsymbol{Z}$. Now, fixing $\boldsymbol{z} \in \mathcal{A}$, we analyze the conditional event $\mathbb{P}[\mathcal{E}(\hat{\boldsymbol{n}})|\boldsymbol{Z}=\boldsymbol{z}]$. By invoking Chernoff's bound

$$\mathbb{P}\left[\mathcal{E}(\hat{\boldsymbol{n}})\middle|\boldsymbol{Z}=\boldsymbol{z}\right] \le \mathbb{E}\left[\exp\left(E_1\left(\boldsymbol{X}', c(\hat{\boldsymbol{n}}), \boldsymbol{z}\right)\right)\right] \quad (38)$$

where

$$E_1\left(\boldsymbol{X}', c(\hat{\boldsymbol{n}}), \boldsymbol{z}\right) = \lambda\|\boldsymbol{z}\|^2 - \lambda\|\boldsymbol{z} + \sqrt{\mathrm{P}' c(\hat{\boldsymbol{n}})}\boldsymbol{X}'\|^2. \quad (39)$$

We choose $\lambda = \lambda^*$, where λ^* is defined in (21). (We make this choice of λ so that the function E_0' defined in (23), and which comes up in our final expression in (6), is maximized.)

Next, utilizing the identity

$$\mathbb{E}\left[e^{-\theta\|\sqrt{\alpha}\boldsymbol{Z}+\boldsymbol{v}\|^2}\right] = \frac{\exp\left(-\frac{\theta\|\boldsymbol{v}\|^2}{1+2\alpha\theta}\right)}{(1+2\alpha\theta)^{\frac{\mathrm{N}}{2}}} \quad (40)$$

which holds for $\alpha > 0$, $\mathbf{v} \in \mathbb{R}^{\mathrm{N}}$, and $2\alpha\theta > -1$, we obtain

$$\mathbb{P}\left[\mathcal{E}(\hat{\boldsymbol{n}})\middle|\boldsymbol{Z}=\boldsymbol{z}\right] \le \exp(f(c(\hat{\boldsymbol{n}}))) \quad (41)$$

where $f(\cdot)$ is defined in (25). Note that the condition $2\alpha\theta > -1$ is ensured in our setup, since $\lambda^* > 0$, and $c(\hat{\boldsymbol{n}}) > 0$. Now, invoking Lemma 4, we conclude that $c(\hat{\boldsymbol{n}}) \ge c_{\min}$. Furthermore, since $\boldsymbol{z} \in \mathcal{A}$, it follows from Lemma 3 that $f(c(\hat{\boldsymbol{n}})) \le f(c_{\min})$. That is, for $\boldsymbol{z} \in \mathcal{A}$, we have

$$\mathbb{P}\left[\mathcal{E}(\hat{\boldsymbol{n}})\middle|\boldsymbol{Z}=\boldsymbol{z}\right] \le e^{\lambda^* f(c_{\min})}. \quad (42)$$

Now, we proceed by applying Gallager's ρ-trick. We fix ℓ, $\mathcal{S}$, i, $\mathcal{N}$, $\hat{\mathcal{S}}$, $\hat{\mathcal{N}}$, and j as in the statement of Lemma 4. For convenience, we set $\eta = (t, \ell, \mathcal{S}, i, \mathcal{N}, j)$, and denote by $\sum_\eta$ and $\bigcup_\eta$ the multiple summations and unions over the indices, respectively. Let $c_1(\hat{\boldsymbol{n}}) = \|\boldsymbol{n}_{\mathcal{S}}\|_1 + \|\boldsymbol{n}_{\mathcal{N}}\|_1 - \|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1$, $c_2(\hat{\boldsymbol{n}}) = \|\hat{\boldsymbol{n}}_{\hat{\mathcal{N}}}\|_1 - \|\boldsymbol{n}_{\hat{\mathcal{N}}}\|_1 + j$, and

$$\hat{\Pi}'(\boldsymbol{n}) = \left\{\hat{\boldsymbol{n}} : c_1(\hat{\boldsymbol{n}}) + c_2(\hat{\boldsymbol{n}}) = 2t\right\} \quad (43)$$

and note that $\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}_t(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}}) = \bigcup_\eta \bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}})$. Fix $p(\boldsymbol{z}, \hat{\boldsymbol{n}}, \mathcal{A}) = \mathbb{1}\{\boldsymbol{z} \in \mathcal{A}\}\mathbb{P}[\mathcal{E}(\hat{\boldsymbol{n}})|\boldsymbol{z}]$. Then, Gallager's ρ-trick applied to (36) yields

$$\mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \Pi_t(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}}) \cap \mathcal{A}\middle|\boldsymbol{z}\right] \le \sum_\eta \mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}}) \cap \mathcal{A}\middle|\boldsymbol{z}\right] \quad (44)$$

$$\le \sum_\eta \left(\sum_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} \mathbb{P}\left[\mathcal{E}(\hat{\boldsymbol{n}}) \cap \mathcal{A}\middle|\boldsymbol{z}\right]\right)^\rho \quad (45)$$

$$= \sum_\eta \left(\sum_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} p(\boldsymbol{z}, \hat{\boldsymbol{n}}, \mathcal{A})\right)^\rho \quad (46)$$

$$\le \sum_\eta \left(\sum_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} e^{f(c_{\min})}\right)^\rho \quad (47)$$

$$\le \sum_\eta |\hat{\Pi}'(\boldsymbol{n})|^\rho e^{\rho f(c_{\min})}. \quad (48)$$

Here, (48) follows because upon fixing η, the parameter $c_{\min}$ is the same for all $\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})$. This, in turn, follows from the definition of $c_{\min}$ in (22), and the observation that, since $\|\boldsymbol{n}\|_1 = \|\hat{\boldsymbol{n}}\|_1$, we have $c_1(\hat{\boldsymbol{n}}) = c_2(\hat{\boldsymbol{n}}) = t$. Using the definition of $f(\cdot)$ in (25), we have

$$\mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}}) \cap \mathcal{A}\middle|\boldsymbol{Z}=\boldsymbol{z}\right] \le |\hat{\Pi}'(\boldsymbol{n})|^\rho e^{\rho b\|\boldsymbol{z}\|^2 - \mathrm{N}\rho a} \quad (49)$$

where a and b are defined as in (19) and (20), respectively.

Now, we take the expectation over $\boldsymbol{Z}$. We observe that, with our choice of λ^*, we have that $1 - 2b\rho > 0$. Then, using (40), we conclude that

$$\mathbb{P}\left[\bigcup_{\hat{\boldsymbol{n}} \in \hat{\Pi}'(\boldsymbol{n})} \mathcal{E}(\hat{\boldsymbol{n}})\right] \le |\hat{\Pi}'(\boldsymbol{n})|^\rho e^{-\frac{\mathrm{N}\rho}{2}\log(1-2b\rho) - \mathrm{N}\rho a}. \quad (50)$$

From Lemma 4, it follows that $|\hat{\Pi}'\boldsymbol{n})| \le \mathrm{M}'$, where M' is given in (16). Using the definition of $\tilde{p}_t$, R, and E_0 in (11)–(15), we conclude that $p_t \le \tilde{p}_t$. □

IV. Numerical Results

In this section, we compute the bound in (6) numerically and compare it with the performance of a TUMA-adapted version of the CCS-AMP algorithm, originally proposed in [8] for the UMA scenario.

The original CCS-AMP scheme comprises of a divide-and-conquer strategy in which messages are split into smaller blocks, and the use of inner and outer encoders and decoders [11]. In particular, the inner decoder reconstructs the transmitted signals, and the outer decoder maps these reconstructions to valid codewords. To extend CCS-AMP to TUMA, we modify the inner decoder to handle message collisions by incorporating a carefully chosen prior and a Bayesian estimation of multiplicities. In the outer decoder, we apply a majority-vote approach to estimate the message multiplicities. With these adaptations, we are able to use CCS-AMP to estimate multiplicities in TUMA.

In the numerical evaluation, we set the codeword length to $n = 38\,400$, the number of message bits to $k = 128$. To use Theorem 1 we need to specify the message type. We choose it as follows. We select message types that approximate a Zipf PMF $p_Z(m; \mathrm{N}_Z, s) = m^{-s}/\sum_{j=1}^{\mathrm{N}_Z} j^{-s}$ for $m = 1, \ldots, \mathrm{N}_Z$. To determine the transmitted message type, we use $p_Z(m; \mathrm{M}_\mathrm{a}, 1)$. Specifically, we fix a K_a' and calculate each n_m by rounding $\mathrm{K}_\mathrm{a}' p_Z(m; \mathrm{M}_\mathrm{a}, 1)$ so that the total number of users $\mathrm{K}_\mathrm{a} = \sum_m n_m$ is approximately K_a'. The resulting transmitted type is $\boldsymbol{t}_Z = \boldsymbol{n}/\mathrm{K}_\mathrm{a}$. For reference, the UMA multiplicity vector is $\boldsymbol{t}_U = [1/\mathrm{K}_\mathrm{a}', \ldots, 1/\mathrm{K}_\mathrm{a}']$. When evaluating the bound, we fix $\delta = 1/(1 - 2\delta_{\min}^*) - 1.01$, and we optimize over ρ, with $\rho_{\min} = 0.01$.

We evaluate the minimum energy per bit $E_\mathrm{b}/N_0 = 10\log_{10}(\mathrm{NP}/2k)$ (dB) required to achieve a total variation distance of $\epsilon = 0.05$. In Fig. 1, we depict the value of E_b/N_0 (dB) versus the total variation distance between the UMA profile and the chosen TUMA profiles, for $\mathrm{K}'_\mathrm{a} = 100$ and M_a varying from 2 to 100. The point $\mathrm{M}_\mathrm{a} = 100$, which corresponds to the UMA setting, is obtained by using [5, Theorem 1]. In our implementation of the CCS-AMP scheme for TUMA, we average the total variation distance over 1000 simulations, and we choose the number of potential candidates chosen per sub-blocks to be 300.

The trend in E_b/N_0—which increases, peaks, and then decreases—can be explained as follows. Assume that the decoder has an estimate of the support of the transmitted messages. Then, it has to consider $\binom{\mathrm{K_a}-1}{\mathrm{M_a}-1}$ possible multiplicity vectors over that support. This number grows, peaks, and then declines as M_a decreases from $\mathrm{K_a}'$ to 1, dictating the complexity in the decoding process. This complexity mirrors the observed trend in E_b/N_0. Furthermore, as we deviate from the UMA profile by decreasing M_a, the power gain per transmitted message increases, eventually peaking at $\mathrm{M}_\mathrm{a} = 1$, where it becomes proportional to $(\mathrm{K}'_\mathrm{a})^2$. This power gain accounts for the eventual substantial improvement[1] in performance, compared to the UMA case when M_a decreases. We also observe that the the E_b/N_0 required by CCS-AMP is about 3dB above the limit predicted by the bound.

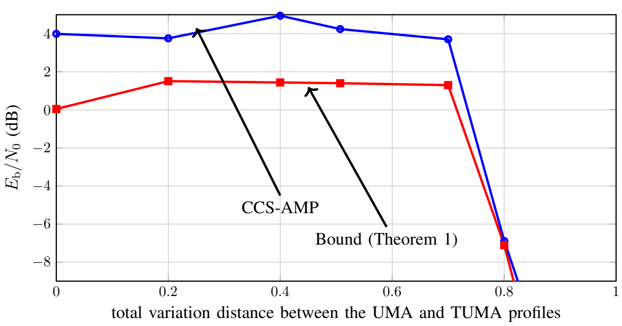

Fig. 1. The required E_b/N_0 (dB) for message types with $(\mathrm{M_a}, \mathrm{K_a})$ in $\{(100, 100), (80, 92), (60, 98), (50, 102), (30, 102), (10, 100)\}$.

V. Conclusion

We derived a numerically computable achievability bound on the error incurred in estimating the type of messages over an unsourced GMAC. By evaluating this bound, we obtained insights into the minimum energy per bit required to transmit a given message type under a total variation constraint. Extending this work to a scenario with unknown user and message counts, as well as to fading models, are possible directions for future works.

Appendix A
Proof of Lemma 1

Since $\mathbb{TV}\big(\boldsymbol{t}, \hat{\boldsymbol{T}}\big) \in [0, 1]$, $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 \in [0, 2\mathrm{K_a}]$. We have,

$$\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = \sum_{m: n_m \geq \hat{n}_m} (n_m - \hat{n}_m) + \sum_{m: \hat{n}_m > n_m} (\hat{n}_m - n_m). \tag{51}$$

Since $\|\boldsymbol{n}\|_1 = \|\hat{\boldsymbol{n}}\|_1$, we have

$$\sum_{m: n_m \geq \hat{n}_m} n_m + \sum_{m: \hat{n}_m > n_m} n_m = \sum_{m: n_m \geq \hat{n}_m} \hat{n}_m + \sum_{m: \hat{n}_m > n_m} \hat{n}_m. \tag{52}$$

Rearranging the terms in the above equation, we observe

$$\sum_{m: n_m \geq \hat{n}_m} (n_m - \hat{n}_m) = \sum_{m: \hat{n}_m > n_m} (\hat{n}_m - n_m). \tag{53}$$

Thus, we observe that $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1$ is even.

Now, we look at the range of values $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1$ can assume. The upper bound on $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1$ is achieved by any pair of $\boldsymbol{n}$ and $\hat{\boldsymbol{n}}$ such that $\mathsf{Supp}(\boldsymbol{n}) \cap \mathsf{Supp}(\hat{\boldsymbol{n}}) = \emptyset$, as

$$\sum_{m: n_m \geq \hat{n}_m} (n_m - \hat{n}_m) = \sum_{m: \hat{n}_m > n_m} (\hat{n}_m - n_m) = \mathrm{K_a}. \tag{54}$$

Next, since we assume $\boldsymbol{n} \neq \hat{\boldsymbol{n}}$, $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 > 0$. Since $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1$ is even, we have $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 \geq 2$. The following choices of $\boldsymbol{n}$ and $\hat{\boldsymbol{n}}$ achieves this lower bound. First, fix any $\boldsymbol{n}$ that satisfy the hypothesis of this lemma. Let m_max denote its index with the maximum multiplicity and m_min the index with minimum multiplicity. Choose an $\hat{\boldsymbol{n}}$ such that $\mathsf{Supp}(\boldsymbol{n}) = \mathsf{Supp}(\hat{\boldsymbol{n}})$, $\hat{n}_{m_\mathrm{max}} = n_{m_\mathrm{max}} - 1$, $\hat{n}_{m_\mathrm{min}} = n_{m_\mathrm{min}} + 1$, and $\hat{n}_m = n_m$, for all other values of $m \in [\mathrm{M}]$. For such a choice, $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2$. Thus, we have shown that, for any $\boldsymbol{n}$ and $\hat{\boldsymbol{n}}$ satisfying the conditions of the lemma, $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1$ is even and belongs to $[2 : 2\mathrm{K_a}]$. Alternatively, $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$, for some $t \in [\mathrm{K_a}]$.

Appendix B
Proof of Lemma 2

Let

$$a' = \frac{1}{2} \log(1 + 2c\mathrm{P}'\lambda) \tag{55}$$

$$b' = \lambda \Big(1 - \frac{1}{1 + 2c\mathrm{P}'\lambda}\Big) \tag{56}$$

so that

$$E'_0(\lambda) = \rho a' + \frac{1}{2} \log(1 - 2b'\rho). \tag{57}$$

To find the λ that maximizes E'_0, we analyze the equation $\frac{dE'_0}{d\lambda} = 0$ and find the *stationary point* of the function E'_0. That is, we analyze

$$\rho \frac{da'}{d\lambda} = \frac{\rho}{1 - 2b'\rho} \frac{db'}{d\lambda}. \tag{58}$$

We have

$$\frac{da'}{d\lambda} = \frac{c\mathrm{P}'}{1 + 2c\mathrm{P}'\lambda} \tag{59}$$

and

$$\frac{db'}{d\lambda} = \frac{4c^2\mathrm{P}'^2\lambda^2 + 4c\mathrm{P}'\lambda}{(1 + 2c\mathrm{P}'\lambda)^2}. \tag{60}$$

[1]Note that when the total variation between the UMA and TUMA profiles is 0.98, the required E_b/N_0 (not shown in the figure) is −27.32 dB for the bound, and −22.46 for CCS-AMP.

We also have

$$1-2b'\rho=\frac{1+2c\mathrm{P}'\lambda-4c\mathrm{P}'\lambda^2\rho}{1+2c\mathrm{P}'\lambda}. \tag{61}$$

Substituting (59), (60), and (61) in (58), we obtain that the stationary point satisfies

$$c\mathrm{P}'+2c^2\mathrm{P}'^2\lambda-4c^2\mathrm{P}'^2\lambda^2\rho=4c^2\mathrm{P}'^2\lambda^2+4c\mathrm{P}'\lambda. \tag{62}$$

Equivalently,

$$4c\mathrm{P}'(1+\rho)\lambda^2+2(2-c\mathrm{P}')\lambda-1=0. \tag{63}$$

Solving the quadratic equation, we obtain (21). From the expression, using the fact that $\rho\in(0,1]$, we can easily observe that

$$\lambda^*\le\frac{1}{2(1+\rho)}<\frac{1}{2}. \tag{64}$$

The claim that λ^* maximizes $E_0'(\lambda)$ follows by observing that $E_0'(\lambda)$ is twice differentiable for $\lambda>0$, and verifying that $\frac{dE_0'(\lambda)}{d\lambda}$ evaluated at $\lambda=\lambda^*$ is negative.

Appendix C
Proof of Lemma 3

We observe that the first derivative of $f(c)$ with respect to c is

$$f'(c)=\frac{\lambda^*\mathrm{P}'}{(1+2c\mathrm{P}'\lambda^*)}\Big(\frac{2\lambda^*\|\boldsymbol{z}\|^2}{1+2c\mathrm{P}'\lambda^*}-\mathrm{N}\Big). \tag{65}$$

Further, $f'(c)=0$ is attained at

$$c^*=\frac{\|\boldsymbol{z}\|^2}{\mathrm{NP}'}-\frac{1}{2\mathrm{P}'\lambda^*}. \tag{66}$$

For any $\boldsymbol{z}\le\mathrm{N}(1+\delta)$, i.e., $\boldsymbol{z}\in\mathcal{A}$, we observe that

$$c^*\le\frac{1+\delta}{\mathrm{P}'}-\frac{1}{2\mathrm{P}'\lambda^*}. \tag{67}$$

From Lemma 2, our choice of λ^* satisfies $\lambda^*=1/2-\delta^*$, for some specific $\delta^*\in(0,1/2)$. Further, the δ^* so obtained is a function of $c_{\min}$, which in turn depends on the parameters t, $\mathcal{S}$, $\mathcal{N}$, ℓ, i, and j. We choose $\delta^*_{\min}$ as the minimum among all such δ^* values. We have $\delta^*_{\min}\in(0,1/2)$ (as we are taking the minimum over a finite set of values). We choose δ such that

$$\delta<\frac{1}{1-2\delta^*_{\min}}-1. \tag{68}$$

This will ensure

$$\frac{1+\delta}{\mathrm{P}'}-\frac{1}{2\mathrm{P}'\lambda^*}<0. \tag{69}$$

Using δ as in (69), with $\boldsymbol{z}\in\mathcal{A}$, and invoking (68), we obtain

$$c^*\le 0. \tag{70}$$

It can be easily verified that, for $\boldsymbol{z}\in\mathcal{A}$,

$$c>c^*\Rightarrow\frac{2\lambda^*\|\boldsymbol{z}\|^2}{1+2c\mathrm{P}'\lambda^*}<\mathrm{N}. \tag{71}$$

Thus, from the expression for $f'(c)$ in (65), it follows that,

$$c>c^*\Rightarrow f'(c)<0. \tag{72}$$

Since $c^*\le 0$ for $\boldsymbol{z}\in\mathcal{A}$, $f(c)$ is strictly decreasing in c, for $c>0$.

Appendix D
Proof of Lemma 4

To establish the bound stated in the first claim of the lemma, we need to enumerate all possible solutions $\hat{\boldsymbol{n}}$ of the nonlinear equation $\|\boldsymbol{n}-\hat{\boldsymbol{n}}\|_1=2t$. We accomplish this by first showing that $\|\boldsymbol{n}-\hat{\boldsymbol{n}}\|_1$ can be expressed as the sum of two terms which are linear in $\hat{\boldsymbol{n}}$, for any $\hat{\boldsymbol{n}}$ stated in the lemma. Then, we obtain the bound by leveraging the resulting linear structure and using the expressions for the total number of positive and non-negative integer-valued solutions of an equation of the form $x_1+\ldots+x_k=n$, for $k,\ n\in\mathbb{N}$ such that $k\le n$.

Fix $\mathcal{S}$, $\mathcal{N}$, $\widehat{\mathcal{N}}$, and $\widehat{\mathcal{S}}$ as in the statement of the lemma and denote $\mathcal{M}_{\mathrm{a}}=\mathtt{Supp}(\boldsymbol{n})$, and $\widehat{\mathcal{M}}_{\mathrm{a}}=\mathtt{Supp}(\hat{\boldsymbol{n}})$. Observe that

$$\begin{aligned}
\|\boldsymbol{n}-\hat{\boldsymbol{n}}\|_1 &= \sum_{\mathrm{m}=1}^{\mathrm{M}}|n_m-\hat{n}_m| && (73)\\
&= \sum_{m\in\mathcal{M}_{\mathrm{a}}}|n_m-\hat{n}_m|+\sum_{m\in\mathcal{M}_{\mathrm{a}}^c}|n_m-\hat{n}_m| && (74)\\
&= \sum_{m\in\mathcal{S}}|n_m-\hat{n}_m|+\sum_{m\in\mathcal{M}_{\mathrm{a}}\cap\widehat{\mathcal{M}}_{\mathrm{a}}}|n_m-\hat{n}_m| \\
&\quad+\sum_{m\in\widehat{\mathcal{S}}}|n_m-\hat{n}_m|+\sum_{m\in\mathcal{M}_{\mathrm{a}}^c\cap\widehat{\mathcal{M}}_{\mathrm{a}}^c}|n_m-\hat{n}_m| && (75)\\
&= \sum_{m\in\mathcal{S}}n_m+\sum_{m\in\mathcal{M}_{\mathrm{a}}\cap\widehat{\mathcal{M}}_{\mathrm{a}}}|n_m-\hat{n}_m|+\sum_{m\in\widehat{\mathcal{S}}}\hat{n}_m+0 && (76)\\
&= \|\boldsymbol{n}_{\mathcal{S}}\|_1+\sum_{m\in\mathcal{M}_{\mathrm{a}}\cap\widehat{\mathcal{M}}_{\mathrm{a}}}|n_m-\hat{n}_m|+j && (77)\\
&= \|\boldsymbol{n}_{\mathcal{S}}\|_1+\sum_{m\in\mathcal{M}_{\mathrm{a}}\cap\widehat{\mathcal{M}}_{\mathrm{a}}\cap\mathcal{N}}|n_m-\hat{n}_m|+ \\
&\quad+\sum_{m\in\mathcal{M}_{\mathrm{a}}\cap\widehat{\mathcal{M}}_{\mathrm{a}}\cap\mathcal{N}^c}|n_m-\hat{n}_m|+j && (78)\\
&= \|\boldsymbol{n}_{\mathcal{S}}\|_1+\sum_{m\in\mathcal{N}}(n_m-\hat{n}_m)+\sum_{m\in\widehat{\mathcal{N}}}(n_m-\hat{n}_m)+j \\
&= \|\boldsymbol{n}_{\mathcal{S}}\|_1+\|\boldsymbol{n}_{\mathcal{N}}\|_1-\|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1 \\
&\quad+\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{N}}}\|_1-\|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1+j. && (79)
\end{aligned}$$

Next, for $\|\boldsymbol{n}-\hat{\boldsymbol{n}}\|_1=2t$, we observe that

$$\|\boldsymbol{n}_{\mathcal{S}}\|_1+\|\boldsymbol{n}_{\mathcal{N}}\|_1-\|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1=t \tag{80}$$

$$\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{N}}}\|_1-\|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1+j=t. \tag{81}$$

Here, (80) is linear in $\hat{\boldsymbol{n}}_{\mathcal{N}}$ and (81) is linear in $\hat{\boldsymbol{n}}_{\widehat{\mathcal{N}}}$, the equations follow by noting that $\|\boldsymbol{n}\|_1=\|\hat{\boldsymbol{n}}\|_1$, and $\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{S}}}\|_1=j$. In (80), $\mathcal{S}$ corresponds to the set of messages miss-detected at the receiver. Furthermore, $\mathcal{N}$ represents the set of detected-but-deflated messages, as the estimated multiplicities corresponding to the messages in $\mathcal{N}$ are deflated with respect to the transmitted multiplicities over $\mathcal{N}$. Likewise, in (81), $\widehat{\mathcal{N}}$ denotes the set of detected-but-inflated messages by the decoder, as the estimated multiplicities of messages in $\widehat{\mathcal{N}}$ are strictly inflated compared to the corresponding transmitted multiplicities.

Finally, $\widehat{\mathcal{S}}$ denotes the set of impostor messages, i.e., messages not transmitted but detected at the receiver.

Next, fix $\mathcal{M}_{\rm a} \subset [\rm M_a]$ such that $|\mathcal{M}_{\rm a}| = \rm M_a$. Furthermore, fix t, ℓ, i as in the statement of the lemma, any subsets $\mathcal{S}$, $\mathcal{N}$, and $\widehat{\mathcal{N}}$ of $\mathcal{M}_{\rm a}$ such that $\mathcal{M}_{\rm a} = \mathcal{S} \cup \widehat{\mathcal{S}} \cup \widehat{\mathcal{N}}$, and any $\widehat{\mathcal{S}} \subset \mathcal{M}_{\rm a}^c \cap [\rm M]$ such that $|\hat{\mathcal{S}}| = \ell$, and $\|\boldsymbol{n}_{\mathcal{S}}\|_1 \geq t$. Then, any solution $\hat{\boldsymbol{n}}^*_{\mathcal{N}}$ to (80), $\hat{\boldsymbol{n}}^*_{\widehat{\mathcal{N}}}$ (81), and $\hat{\boldsymbol{n}}^*_{\widehat{\mathcal{S}}}$ to $\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{S}}}\|_1 = j$ define a solution $\hat{\boldsymbol{n}}^*_{\mathcal{N}\cup\widehat{\mathcal{N}}\cup\widehat{\mathcal{S}}}$ to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$. Accordingly, fixing t, ℓ, i, and j as mentioned, $\mathcal{S} \subseteq \mathcal{M}_{\rm a}$ such that $|\mathcal{S}| = \ell$, $\|\boldsymbol{n}_{\mathcal{S}}\|_1 \geq t$ $\mathcal{N} \subseteq \mathcal{M}_{\rm a} \cap \mathcal{S}^c$ such that $|\mathcal{N}| = i$, and $\widehat{\mathcal{N}} = \mathcal{M}_{\rm a} \setminus (\mathcal{S} \cup \mathcal{N})$,

$$\bigcup_{\widehat{\mathcal{S}}}\bigcup_{\hat{\boldsymbol{n}}^*_{\widehat{\mathcal{S}}}}\bigcup_{\hat{\boldsymbol{n}}^*_{\mathcal{N}}}\bigcup_{\hat{\boldsymbol{n}}^*_{\widehat{\mathcal{N}}}}\Big\{\hat{\boldsymbol{n}}^*_{\mathcal{N}\cup\widehat{\mathcal{N}}\cup\widehat{\mathcal{S}}} : \|\hat{\boldsymbol{n}}_{\widehat{\mathcal{S}}}\|_1 = j,\ \text{(80), and (81) holds}\Big\} \tag{82}$$

where $\widehat{\mathcal{S}}$ is such that $|\widehat{\mathcal{S}}| = \ell$, contains the set of all solutions to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$. In addition, if $\mathcal{N}$ forms a deflated set for any $\hat{\boldsymbol{n}}$ of this set (with respect of $\boldsymbol{n}$), i.e., if $\mathcal{N} = \{m : n_m \geq \hat{n}_m\}$, (82) represents the set of all solutions to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$.

We observe that there are $\binom{\rm M - M_a}{\ell}$ ways of choosing an impostor set $\widehat{\mathcal{S}}$ of length ℓ in (82), and that

$$\binom{\rm M - M_a}{\ell} \leq \frac{\rm M^\ell}{\ell!}. \tag{83}$$

For any such $\widehat{\mathcal{S}}$, there are

$$\binom{j-1}{\ell-1} \tag{84}$$

number of solutions to $\|\hat{\boldsymbol{n}}_{\widehat{\mathcal{S}}}\|_1 = j$ in $\mathbb{N}$.

Next, note that there are $\rm M^+$ solutions of $\hat{\boldsymbol{n}}$ to (80) in $\mathbb{N}$, where $\rm M^+$ is defined in (18). In addition, there are $\rm M_0^+$ solutions of $\boldsymbol{n} - \hat{\boldsymbol{n}}$ to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = t - \|\boldsymbol{n}_{\mathcal{S}}\|_1$ in $\mathbb{N}_0$, where M_0^+ is defined in (17). Hence, there are at most $\min\{\rm M_0^+, M_0\}$ valid options for the deflated multiplicities in any solution $\hat{\boldsymbol{n}}$ to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$.

Finally, there are

$$\binom{t-j-1}{\rm M_a - \ell - i - 1} \tag{85}$$

solutions of $\hat{\boldsymbol{n}}$ to (81) in $\mathbb{N}$. Combining (83)–(85), we obtain $\rm M'$ in (16) as the number of solutions of $\hat{\boldsymbol{n}}$ to $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|_1 = 2t$.

Next, to prove the second part of the lemma, we utilize the inequality

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2,\ \boldsymbol{x} \in \mathbb{R}^n. \tag{86}$$

We obtain $\|\boldsymbol{n} - \hat{\boldsymbol{n}}\|^2 \geq c_{\min}$, where $c_{\min}$ is as in (22), by applying the above inequality to $\boldsymbol{n}_{\mathcal{S}}$, $\|\boldsymbol{n}_{\mathcal{N}}\|_1 - \|\hat{\boldsymbol{n}}_{\mathcal{N}}\|_1$, $\|\hat{\boldsymbol{n}}_{\hat{\mathcal{N}}}\|_1 - \|\boldsymbol{n}_{\hat{\mathcal{N}}}\|_1$, and $\hat{\boldsymbol{n}}_{\hat{\mathcal{S}}}$, separately.

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