I. Proof for Saddlepoint Approximation on RCUs bound

The proof of the saddlepoint approximation that we will show next follows the steps in [1, App. I.A] and [2, App. E], which in turn are mostly based on [3, Ch. XVI.4, Th.1].

A. Preliminaries

Let $\{Z_\ell\}_{\ell=1}^n$ be a sequence of i.i.d., real-valued, zero-mean random variables. The MGF of Z_ℓ is defined as

$$m(\zeta) = \mathbb{E}\left[e^{\zeta Z_\ell}\right] \tag{1}$$

and the CGF is defined as

$$\gamma(\zeta) = \log m(\zeta). \tag{2}$$

We assume that Z_ℓ is nonlattice. Indeed, in our setup, Z_ℓ is a continuous random variable. For lattice distributions, see [3, Ch. XVI.4, Th.2]. We further assume

$$\sup_{\underline{\zeta}<\zeta<\overline{\zeta}} \left| \frac{d^3}{d\zeta^3} m(\zeta) \right| < \infty. \tag{3}$$

We next show the saddlepoint approximation for a simpler case than the one given by the RCUs bound, whose tail probability also presents a uniform random variable U in $[0,1]$ in the expression. In particular we will show that

$$\mathbb{P}\left[\sum_{\ell=1}^n Z_\ell > \mathrm{R}\right] = e^{n(\gamma(\zeta)-\zeta\gamma'(\zeta))}\left[\Phi_{n,\zeta}(\zeta) + \frac{K(\zeta,\zeta,n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right] \tag{4}$$

and

$$\mathbb{P}\left[\sum_{\ell=1}^n Z_\ell < \mathrm{R}\right] = 1 - e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\left[\Phi_{n,\zeta}(-\zeta) - \frac{K(-\zeta,\zeta,n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right] \tag{5}$$

where

$$K(u,\zeta,n) = \frac{\gamma'''(\zeta)}{6\gamma''(\zeta)^{3/2}}\left(- \frac{1}{\sqrt{2\pi}} + \frac{u^2 n\gamma''(\zeta)}{\sqrt{2\pi}} - u^3(\gamma''(\zeta)n)^{3/2}\Phi_{n,\zeta}(u)\right) \tag{6}$$

and

$$\Phi_{b,\zeta}(u) = e^{b\frac{u^2}{2}\gamma''(\zeta)} Q\left(u\sqrt{b\gamma''(\zeta)}\right). \tag{7}$$

We start with $\mathbb{P}[\sum_{\ell=1}^n Z_\ell > \mathrm{R}]$ for $\mathrm{R} > 0$. Let $Y_\ell = Z_\ell - \tilde{\mathrm{R}}$, where $\tilde{\mathrm{R}} = \mathrm{R}/n$ and let F denote the distribution of Y_ℓ. Then, the CGF of Y_ℓ is given by $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \zeta\tilde{\mathrm{R}}$. Let the tilted random variable V_ℓ have distribution

$$v_\zeta(x) = e^{-\tilde{\gamma}(\zeta)} \int_{-\infty}^{x} e^{\zeta t} dF(t) \tag{8}$$

$$= e^{-\gamma(\zeta)+\zeta\tilde{\mathrm{R}}} \int_{-\infty}^{x} e^{\zeta t} dF(t). \tag{9}$$

Let $\psi_\zeta(\tau)$ denote the MGF of the tilted random variable V_ℓ, which is given by

$$\begin{aligned}\psi_\zeta(\tau) &= \int_{\infty}^{\infty} e^{\tau x} dv_\zeta(x) \\ &= \int_{-\infty}^{\infty} e^{\tau x - \gamma(\zeta)+\zeta\tilde{\mathrm{R}}+\zeta x} dF(x) \\ &= e^{-\gamma(\zeta)+\zeta\tilde{\mathrm{R}}} \int_{-\infty}^{\infty} e^{(\tau+\zeta)x} dF(x) \\ &= e^{-\gamma(\zeta)+\zeta\tilde{\mathrm{R}}}\, \mathbb{E}\left[e^{\tau+\zeta(Z_\ell-\tilde{\mathrm{R}})}\right] \\ &= e^{-\gamma(\zeta)}\, \mathbb{E}\left[e^{(\tau+\zeta)Z_\ell}\right] e^{-\tau\tilde{\mathrm{R}}} \\ &= \frac{m(\tau+\zeta)}{m(\zeta)} e^{-\tau\tilde{\mathrm{R}}}.\end{aligned} \tag{10}$$

Since $\mathbb{E}[V_\ell] = \psi'_\zeta(0)$, where the derivative is taken with respect to τ, it follows that

$$\begin{aligned}\mathbb{E}[V_\ell] &= \psi'_\zeta(0) \\ &= \left(\frac{m'(\tau+\zeta)}{m(\zeta)} e^{-\tau\tilde{\mathrm{R}}} - \tilde{\mathrm{R}}\frac{m(\tau+\zeta)}{m(\zeta)} e^{-\tau\tilde{\mathrm{R}}}\right)\Bigg|_{\tau=0} \\ &= \frac{m'(\zeta)}{m(\zeta)} - \tilde{\mathrm{R}} \\ &= \gamma'(\zeta) - \tilde{\mathrm{R}}.\end{aligned} \tag{11}$$

Similarly, $\mathbb{V}\mathrm{ar}[V_\ell] = \mathbb{E}\left[V_\ell^2\right] - \mathbb{E}[V_\ell]^2 = \gamma''(\zeta)$.

We denote by F^{*n} the distribution of $\sum_{\ell=1}^n Y_\ell$ and by v_ζ^{*n} the distribution of $\sum_{\ell=1}^n V_\ell$. Proceeding as in (8), we obtain

$$\begin{aligned}v_\zeta^{*n}(x) &= e^{-n\tilde{\gamma}(\zeta)} \int_{-\infty}^{x} e^{\zeta t} dF^{*n}(t) \\ &= e^{-n\gamma(\zeta)+\zeta\tilde{\mathrm{R}}} \int_{-\infty}^{x} e^{\zeta t} dF^{*n}(t).\end{aligned} \tag{12}$$

We next require an expression for $1 - F^{*n}$ as a function of $v_\zeta^{*n}(x)$, which can be obtained by inverting (12) and noting that $\mathbb{P}[\sum_{\ell=1}^n Z_\ell \geq \mathrm{R}] = 1 - F^{*n}(\mathrm{R})$. Thus,

$$\mathbb{P}\left[\sum_{\ell=1}^n Z_\ell \geq R\right] = e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_0^\infty e^{-\zeta y} dv_\zeta^{*n}(y). \tag{13}$$

We next choose ζ such that $n\gamma'(\zeta) = \mathrm{R}$, which ensures that the distribution v_ζ^{*n} has zero mean. We then replace the distribution v_ζ^{*n} by the zero-mean normal distribution with variance $n\gamma''(\zeta)$, denoted by $\mathfrak{N}_{n,\gamma''(\zeta)}$, and analyze the error incurred by this substitution. Let first

$$
\begin{aligned}
A_\zeta &= e^{n\gamma(\zeta)-\zeta R}\int_0^\infty e^{-\zeta y} d\mathfrak{N}_{n,\gamma''(\zeta)}(y)\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi n\gamma''(\zeta)}}\int_0^\infty e^{-\zeta y} e^{-\frac{y^2}{2n\gamma''(\zeta)}} dy\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi}}\int_0^\infty e^{-\zeta t\sqrt{n\gamma''(\zeta)}} e^{-\frac{t^2}{2}} dt\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)]}}{\sqrt{2\pi}}\int_\zeta^\infty e^{-\frac{1}{2}\left(t+\zeta\sqrt{n\gamma''(\zeta)}\right)^2} dt\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)]}}{\sqrt{2\pi}}\int_{\zeta\sqrt{n\gamma''(\zeta)}}^\infty e^{-\frac{x^2}{2}} dx\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)]} Q(\zeta\sqrt{n\gamma''(\zeta)})\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\Phi_{n,\zeta}(\zeta) \qquad (14)
\end{aligned}
$$

where the third equality follows by the change of variable $t = y/\sqrt{n\gamma''(\zeta)}$, and the fifth equality follows by the change of variable $x = t + \zeta\sqrt{n\gamma''(\zeta)}$.

We are now ready to asses the error incurred by replacing v_ζ^{*n} with $\mathfrak{N}_{n,\gamma''(\zeta)}$ in (13), which is given by

$$
\begin{aligned}
&e^{n\gamma(\zeta)-\zeta R}\int_0^\infty e^{-\zeta y} dv_\zeta^{*n}(y) - A_\zeta\\
&\quad= e^{n[\gamma(\zeta)-\zeta\gamma(\zeta)]}\Bigg[-\left(v_\zeta^{*n}(0)-\mathfrak{N}_{n,\gamma''(\zeta)}\right)\\
&\qquad\qquad +\zeta\int_0^\infty \left(v_\zeta^{*n}(y)-\mathfrak{N}_{n,\gamma''(\zeta)}(y)\right)e^{-\zeta y}dy\Bigg]\\
&\quad= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\Bigg[\frac{\gamma'''(\zeta)}{6\gamma''(\zeta)^{3/2}\sqrt{n}}\Bigg(-\frac{1}{\sqrt{2\pi}}\\
&\qquad\qquad +\frac{\zeta^2 n\gamma''(\zeta)}{\sqrt{2\pi}} - \zeta^3\gamma''(\zeta)^{3/2}n^{3/2}\Phi_{n,\zeta}(\zeta)\Bigg) + o\left(\frac{1}{\sqrt{n}}\right)\Bigg]\\
&\quad= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\left(\frac{K(\zeta,\zeta,n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) \qquad (15)
\end{aligned}
$$

where the second equality follows from [3, Sec. XVI.4, Th. 1]. Note that substitution error in (15) converges only when the condition in (3) is met. Combining (13)-(15) with the choice $n\gamma'(\zeta) = \mathrm{R}$, we establish (4).

We next consider the tail probability in (5), i.e., $\mathbb{P}[\sum_{\ell=1}^n Z_\ell < \mathrm{R}]$. Since the proof of the saddlepoint approximation of this tail probability is very similar to the proof of (4), we will only focus on the differences. It follows that

$$
\mathbb{P}\left[\sum_{\ell=1}^n Z_\ell < \mathrm{R}\right] = e^{n\gamma(\zeta)-\zeta \mathrm{R}}\int_{-\infty}^0 e^{-\zeta y} dv_\zeta^{*n}(y). \qquad (16)
$$

We again choose ζ such that $n\gamma'(\zeta) = \mathrm{R}$, and define

$$
\begin{aligned}
\tilde{A}_\zeta &= e^{n\gamma(\zeta)-\zeta \mathrm{R}}\int_{-\infty}^0 e^{-\zeta y} d\mathfrak{N}_{n,\gamma''(\zeta)}(y)\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi n\gamma''(\zeta)}}\int_{-\infty}^0 e^{-\zeta y} e^{-\frac{y^2}{2n\gamma''(\zeta)}} dy\\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-\zeta t\sqrt{n\gamma''(\zeta)}} e^{-t^2/2} dt\\
&= \frac{e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)\right]}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{\frac{1}{2}(t+\zeta\sqrt{n\gamma''(\zeta)})^2} dt\\
&= \frac{e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)\right]}}{\sqrt{2\pi}}\int_{-\infty}^{\zeta\sqrt{n\gamma''(\zeta)}} e^{-\frac{x^2}{2}} dx\\
&= \frac{e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)\right]}}{\sqrt{2\pi}}\int_{-\zeta\sqrt{n\gamma''(\zeta)}}^\infty e^{-\frac{x^2}{2}} dx\\
&= e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)\right]} Q(-\zeta\sqrt{n\gamma''(\zeta)})\\
&= e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)\right]}\Phi_{n,\zeta}(-\zeta) \qquad (17)
\end{aligned}
$$

where the third equality by the change of variable $t = y/\sqrt{n\gamma''(\zeta)}$, and the fifth equality follows by the change of variable $x = t+\zeta\sqrt{n\gamma''(\zeta)}$. The error incurred by substituting v_ζ^{*n} by $\mathfrak{N}_{n,\gamma''(\zeta)}$ is given by

$$
\begin{aligned}
&e^{n\gamma(\zeta)-\zeta \mathrm{R}}\int_{-\infty}^0 e^{-\zeta y} dv_\zeta^{*n}(y) - \tilde{A}_\zeta\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\Bigg[\left(v_\zeta^{n*}(0)-\mathfrak{N}_{n,\gamma''(\zeta)}(0)\right)\\
&\qquad +\zeta\int_{-\infty}^0 \left(v_\zeta^{*n}(y)-\mathfrak{N}_{n,\gamma''(\zeta)}(y)\right)e^{\zeta y}dy\Bigg]\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\Bigg[\frac{1}{\sqrt{2\pi}}\frac{\gamma'''(\zeta)}{6\gamma''(\zeta)^{3/2}\sqrt{n}}\Bigg(1+\int_{-\infty}^0\zeta\\
&\qquad \times\sqrt{\gamma''(\zeta)n}(1-z^2)e^{-\zeta\sqrt{\gamma''(\zeta)n}z-\frac{z^2}{2}}dz\Bigg) + o\left(\frac{1}{\sqrt{n}}\right)\Bigg]\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\Bigg[\frac{\gamma'''(\zeta)}{6\gamma''(\zeta)^{3/2}\sqrt{n}}\Bigg(\frac{1}{\sqrt{2\pi}} - \frac{\zeta^2\gamma''(\zeta)n}{\sqrt{2\pi}}\\
&\qquad -\zeta^3(\gamma''(\zeta)n)^{3/2}\Phi_{n,\zeta}(-\zeta)\Bigg) + o\left(\frac{1}{\sqrt{n}}\right)\Bigg]\\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}\left(-\frac{K(-\zeta,\zeta,n)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right). \qquad (18)
\end{aligned}
$$

By combining this result with $\tilde{A}_\zeta$ in (17) for $n\gamma'(\zeta) = \mathrm{R}$, we establish (5).

B. Extension to the RCUs Bound

In this section, we will show how to obtain the saddlepoint expansion of the tail probability appearing in the RCUs bound, namely, $\mathbb{P}[\sum_{\ell=1}^n Z_\ell \geq \mathrm{R} + \log U]$. Different from the previous section, we now have the term $\log U$, where U is a uniformly distributed random variable in the interval $[0, 1]$. To compute the expansion of this tail probability, we will follow the steps detailed in [1, App. 1-B] and [2, App. E]. We start with the case $R > 0$. If $\zeta \in [0,1]$, our proof coincides with the one in [1, App. 1-B]. Nevertheless, we will reproduce it here for the sake of completeness. It follows that

$$\begin{aligned}
&\mathbb{P}\left[\sum_{\ell=1}^{n} Z_\ell \geq \mathrm{R} + \log U\right] \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_0^1 \int_{\log u}^{\infty} e^{-\zeta y} dv_\zeta^{*n}(y) du \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_{-\infty}^{\infty} \int_0^{\min(1,e^y)} e^{-\zeta y} du dv_\zeta^{*n}(y) \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \left(\int_0^{\infty} e^{-\zeta y} dv_\zeta^{*n}(y) + \int_{-\infty}^{0} e^{(1-\zeta)y} dv_\zeta^{*n}(y) \right).
\end{aligned} \tag{19}$$

The first term in (19) coincides with the A_ζ given in (14). Similarly, to analyze the second term, we first define

$$\begin{aligned}
B_\zeta &= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_{-\infty}^{0} e^{(1-\zeta)y} d\mathfrak{N}_{n,\gamma''(\zeta)}(y) \\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi n\gamma''(\zeta)}} \int_{-\infty}^{0} e^{(1-\zeta)y} e^{-\frac{y^2}{2n\gamma''(\zeta)}} dy \\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]}}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1-\zeta)t\sqrt{n\gamma''(\zeta)}} e^{-\frac{t^2}{2}} dt \\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{(1-\zeta)^2}{2}\gamma''(\zeta)]}}{\sqrt{2\pi}} \int_{\zeta}^{\infty} e^{-\frac{1}{2}\left(t-(1-\zeta)\sqrt{n\gamma''(\zeta)}\right)^2} dt \\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{(1-\zeta)^2}{2}\gamma''(\zeta)]}}{\sqrt{2\pi}} \int_{-\infty}^{-(1-\zeta)\sqrt{n\gamma''(\zeta)}} e^{-\frac{x^2}{2}} dx \\
&= \frac{e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{(1-\zeta)^2}{2}\gamma''(\zeta)]}}{\sqrt{2\pi}} \int_{(1-\zeta)\sqrt{n\gamma''(\zeta)}}^{\infty} e^{-\frac{x^2}{2}} dx \\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{(1-\zeta)^2}{2}\gamma''(\zeta)]} Q\left((1-\zeta)\sqrt{n\gamma''(\zeta)}\right) \\
&= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]} \Phi_{n,\zeta}(1-\zeta).
\end{aligned} \tag{20}$$

The third equality follows by the change of variable $t = y/\sqrt{n\gamma''(\zeta)}$ and the fourth equality follows by the change of variable $x = t - (1-\zeta)\sqrt{n\gamma''(\zeta)}$. By following steps similar to (14)-(15) (where we studied the error incurred by replacing v_ζ^{*n} with $\mathfrak{N}_{n,\gamma''(\zeta)}$) also with B_ζ, after some mathematical manipulations, it follows that

$$\begin{aligned}
\mathbb{P}\left[\sum_{\ell=1}^{n} Z_\ell \geq \mathrm{R} + \log U\right] &= e^{n[\gamma(\zeta)-\zeta\gamma'(\zeta)]} \\
&\times \left[\Phi_{n,\zeta}(\zeta) + \Phi_{n,\zeta}(1-\zeta) + o\left(\frac{1}{\sqrt{n}}\right)\right]
\end{aligned} \tag{21}$$

which concludes the proof for $\zeta[0,1]$. It can be shown that (20) tends to infinity as $n \to \infty$ when $\zeta > 1$.[1] To address the case $\zeta > 1$, we start with (19) and instead of making the choice of ζ such that $n\gamma'(\zeta) = \mathrm{R}$, we choose $\zeta = 1$. As a consequence, we now need to analyze the error incurred by replacing v_ζ^{*n} with the normal distribution that has mean $n\gamma'(\zeta) - \mathrm{R}$ and variance $n\gamma''(\zeta)$, denoted by $\tilde{\mathfrak{N}}_{n,\gamma''(\zeta)}$. We next expand the first integral in (20), which we denote by

$$\begin{aligned}
C_\zeta &= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_0^{\infty} e^{-\zeta y} d\tilde{\mathfrak{N}}_{n,\gamma''(\zeta)}(y) \\
&= \frac{e^{n\gamma(\zeta)-\zeta\mathrm{R}}}{\sqrt{2\pi n\gamma''(\zeta)}} \int_0^{\infty} e^{-\zeta y} e^{-\frac{(y-n\gamma'(\zeta)+\mathrm{R})^2}{2n\gamma''(\zeta)}} dy \\
&= e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)+\frac{\zeta^2}{2}\gamma''(\zeta)\right]} Q\left(\frac{\gamma - n\gamma'(\zeta)}{\sqrt{n\gamma''(\zeta)}} + \zeta\sqrt{n\gamma''(\zeta)}\right) \\
&= e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)\right]} \tilde{\Phi}_n(\zeta,\zeta)
\end{aligned} \tag{22}$$

where the third equality follows by the change of variables $y = t\sqrt{n\gamma''(\zeta)} + n\gamma'(\zeta) - \mathrm{R}$ and $x = t + \zeta\sqrt{n\gamma''(\zeta)}$, and where

$$\begin{aligned}
\tilde{\Phi}_b(a_1,a_2) &= e^{ba_1\left[-\gamma'(1)-\mathrm{R}+\frac{\gamma''(1)}{2}\right]} \\
&\times Q\left(a_1\sqrt{b\gamma''(1)} - a_2\frac{b(\gamma'(1)+\mathrm{R})}{\sqrt{b\gamma''(1)}}\right).
\end{aligned} \tag{23}$$

By following the same steps (with a slightly different change of variables), we next expand the second integral in (20), which is denoted by

$$\begin{aligned}
\tilde{C}_{(\zeta)} &= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_{-\infty}^{0} e^{(1-\zeta)y} d\tilde{\mathfrak{N}}_{n,\gamma''(\zeta)}(y) \\
&= \frac{e^{n\gamma(\zeta)-\zeta\mathrm{R}}}{\sqrt{2\pi\gamma''(\zeta)}} \int_{-\infty}^{0} e^{(1-\zeta)y} e^{-\frac{(y-n\gamma'(\zeta)+\mathrm{R})^2}{2n\gamma''(\zeta)}} dy \\
&= e^{n\gamma(\zeta)-\zeta\gamma'(\zeta)} \left[\tilde{\Phi}_n(1-\zeta,-\zeta)\right].
\end{aligned} \tag{24}$$

Proceeding as in (14)-(15) with C_ζ and $\tilde{C}_\zeta$ particularized for $\zeta = 1$,[2] it follows that for $\zeta > 1$ and $\mathrm{R} > 0$,

$$\begin{aligned}
\mathbb{P}\left[\sum_{\ell=1}^{n} Z_\ell \geq \mathrm{R} + \log U\right] &= e^{n\gamma(1)-\mathrm{R}}\Big[\tilde{\Phi}_n(1,1) \\
&+ \tilde{\Phi}_n(0,-1) + o\left(\frac{1}{\sqrt{n}}\right)\Big].
\end{aligned} \tag{25}$$

It only remains to show the saddlepoint expansion of the tail probability $P[\sum_{\ell=1}^{n} Z_\ell \geq \mathrm{R} + \log U] = 1 - P[\sum_{\ell=1}^{n} Z_\ell < \mathrm{R} + \log U]$ when $\mathrm{R} < 0$, in which case the choice $n\gamma'(\zeta) = \mathrm{R}$ yields $\zeta < 0$. In this case, it follows that

$$\begin{aligned}
&\mathbb{P}\left[\sum_{\ell=1}^{n} Z_\ell < \mathrm{R} + \log U\right] \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_0^1 \int_{-\infty}^{\log u} e^{-\zeta y} dv_\zeta^{*n}(y) du \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \int_{-\infty}^{0} \int_{e^y}^{1} e^{-\zeta y} du dv_\zeta^{*n}(y) \\
&= e^{n\gamma(\zeta)-\zeta\mathrm{R}} \left(\int_{-\infty}^{0} e^{-\zeta y} dv_\zeta^{*n}(y) + \int_{-\infty}^{0} e^{(1-\zeta)y} dv_\zeta^{*n}(y) \right).
\end{aligned} \tag{26}$$

[1]This can be seen when analyzing the error incurred by substituting v_ζ^{*n} by the normal distribution, and expanding the expression similar to (14)-(15).

[2]The choice of $\zeta = 1$ ensures that the exponential term $e^{(1-\zeta)y}$ in (24) does not go to infinity as $n \to \infty$.

Here, the first integral coincides with $\tilde{A}_\zeta$ and the second integral coincides with B_ζ. Thus, proceeding as in (14)-(15), it can be shown that

$$P\left[\sum_{\ell=1}^{n} Z_\ell \geq \mathrm{R} + \log U\right] = 1 - e^{n\left[\gamma(\zeta)-\zeta\gamma'(\zeta)\right]}\Bigg[\Phi_{n,\zeta}(-\zeta) - \Phi_{n,\zeta}(1-\zeta) + o\left(\frac{1}{\sqrt{n}}\right)\Bigg] \tag{27}$$

which concludes the proof of the saddlepoint approximation of the RCUs bound.

References

[1] A. Lancho, J. Östman, G. Durisi, T. Koch, and G. Vazquez-Vilar, "Saddlepoint approximations for short-packet wireless communications," *IEEE Trans. Wireless Commun.*, vol. 19, no. 7, pp. 4831–4846, Jul. 2020.

[2] J. Scarlett, A. Martinez, and A. Guillén i Fàbregas, "Mismatched decoding: Error exponents, second-order rates and saddlepoint approximations," *IEEE Trans. Inf. Theory*, vol. 60, no. 5, pp. 2647–2666, May 2014.

[3] W. Feller, *An introduction to probability theory and its applications*. Wiley, 1971, vol. 2.